Journal of Applied Mechanics and Technical Physics, Vol. 51, No. 3, pp. 437–447, 2010

FORMATION OF A CONTACT ZONE DURING COMPRESSION OF A PLATE WITH AN ELLIPTIC HOLE

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UDC 539.74375

The problem of compression of a thin plate with an elliptic hole is considered. It is assumed that increasing the distant compressive load can lead to contact of opposite regions of the boundaries of the ellipse. The problem is solved within the framework of a modified Leonov–Panasyuk–Dugdale model and an elastoplastic analog of the Griffith problem for an ellipse using the Goodier and Kanninen model. The critical fracture parameters providing an equilibrium configuration of the system are determined from a sufficient strength criterion representing a system of two equations, one of which specifies the absence of partial overlapping of the upper and lower surfaces of the contact zone, and the other is a deformation criterion of critical opening of the ellipse. The compression-induced deformation of the boundaries of ellipses with various curvature radii at the top is shown by the example of annealed copper having nanostructure.

Key words: *ellipse, contact zone, prefracture zone, modified Leonov–Panasyuk–Dugdale model, elastoplastic analog of Griffith problem, Goodier and Kanninen model, critical fracture parameters.*

INTRODUCTION

Compression of a thin plate of an elastoplastic a material with an elliptic hole by homogeneous stresses at infinity applied perpendicular to the major axis of the hole leads to a change in the geometrical shape of the hole. During compression, plastic zones are formed ahead of the tops of the ellipse. External stresses can cause contact of the opposite surfaces of the ellipse. The presence of a contact zone in a rounded-tip crack in an elastoplastic plate under compression was suggested in [1].

We consider a model in which the contact area is a region of interaction between two layers of atoms arranged symmetrically about the major axis of the deformed ellipse. The initial stage of the closing process is characterized by the occurrence of an attraction force for the atomic pair at the center of the deformed ellipse. An increase in the distant compressive forces leads to an increase in the number of atomic pairs approaching each other on both sides of the center. The interaction of each atomic pair, except for the two pairs at the ends of the contact area, includes the attraction, equilibrium, and repulsion stages. According to Hook's law [2], the atomic repulsion force increases in proportion to the distance of approach of the atomic layers. With further approach, this force reaches a maximum at the center of the deformed ellipse. In approaching the ends of the contact area, the atomic attraction force decreases to zero. The relationship between the atomic attraction force and the distance of approach of the atomic planes is nonlinear. To avoid solving the nonlinear problem, we use the Goodier and Kanninen model for the case of tension [2]. We consider a modification of this model for the case of compression. In the modified Goodier and Kanninen models for the case of compression, the two layers of atoms forming the contact area divide the plate into two half. The material above and below the contact area is considered linearly elastic. A diagram of

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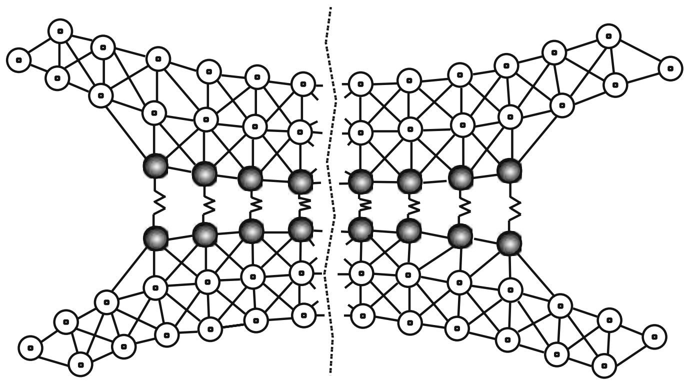

Fig. 1. Contact zone of a deformed elliptic hole during plate compression.

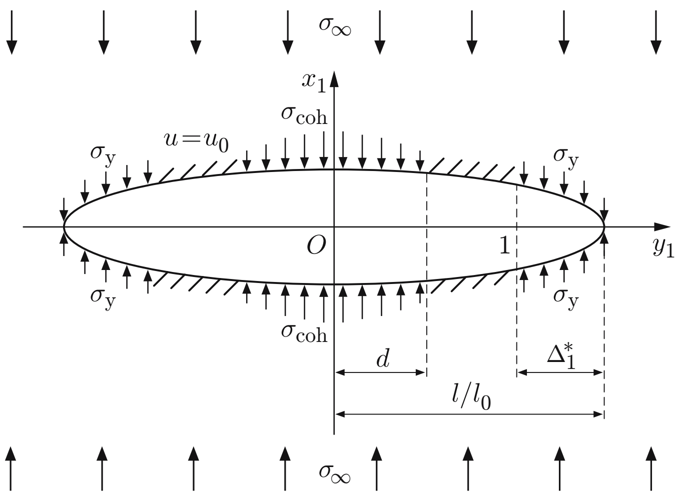

Fig. 2. Diagram of compression of a plate with an elliptic hole leading to the formation of a contact zone.

the approach of the atomic layers is shown in Fig. 1. The atoms of the two layers in the process of approach are shaded. The straight and broken lines (spring) connecting the atoms correspond to the linear and nonlinear laws of interaction of the atoms. For simplicity, only straight-line connections are included in the model. Thus, in Fig. 1, the contact area is represented as the region of interaction of two layers of atoms. Taking into account the complex nature of interaction of the atomic layers, we next assume that the force of interaction of the atomic pairs depends on their approach and obeys a sinusoidal, exponential (with a negative power) or linear law [2].

1. FORMULATION OF THE PROBLEM

A thin plate with an elliptic hole is compressed at infinity by homogeneous stresses σ_∞ acting along the normal to the major axis of the ellipse of length $2l_0$ located along the Oy_1 axis (Fig. 2). The origin of the coordinate system Ox_1y_1 ($x_1 = x/l_0$, $y_1 = y/l_0$) is at the center of the ellipse. Compression of the plate leads to the formation of plastic zones in the vicinities of the tops of the ellipse. It is assumed that an increase in the distant compressive load can lead to contact of the opposite regions of the boundaries of the ellipse. Let $2d$ be the length of the contact zone. In this case, the presence of plastic zones prevents the complete closure of the opposite regions of the boundaries of the deformed ellipse when $d = l_0$.

Let us determine the critical fracture parameters: the critical external compressive stress σ_∞^* and critical lengths of the contact zone $2d^*$ and the prefracture zone Δ^* at which the plate with the deformed ellipse is in dynamic equilibrium provided that there is no partial overlapping of the surfaces in the contact zone.

To solve the problem, we use a modified Leonov–Panasyuk–Dugdale model and an elastoplastic analog of the Griffith problem [3]. Within the framework of the modified Leonov–Panasyuk–Dugdale model for compression, the real deformed ellipse is replaced with a fictitious deformed ellipse. The lengths of the major axes of the real and fictitious deformed ellipses are equal to $2l_0$ and $2(l_0 + \Delta) = 2l$, respectively. We note that, at $x = 0$, the length of the prefracture zone Δ cannot coincide with the length of the plastic zone. The end regions of the fictitious deformed ellipse ($l_0 < |y| \leqslant l$) are filled with a plastically deformed material, whose cohesion is replaced with a constant stress equal to the yield point of the material σ_{y}. In the contact zone ($|y| \leqslant d$, $d < l_0$), the opposite regions of the boundaries are attracted to each other by cohesion forces. The regions of the boundary of the deformed ellipse $d < |y| < l_0$ are free of stresses.

To reduce the elastoplastic problem defined for the modified Leonov–Panasyuk–Dugdale model to a boundary-value elastic problem, we use a modified elastoplastic analog of the Griffith problem for an ellipse in which a thin plate with a fictitious ellipse is compressed by constant stresses at infinity σ_∞ (see Fig. 2). The end regions of the fictitious ellipse $l_0 \leqslant |y| \leqslant l$ are subjected to compressive stresses $-\sigma_{\text{y}}$ by which is meant the material cohesion. We shall denote the cohesion stresses of the material by σ_{coh}. The regions of the boundaries of the fictitious ellipse $|y| \leqslant d$ are subjected to compressive stresses, resulting in the formation of the contact zone. As noted above, atomic interaction forces act in the contact zone. To take into account the reduction in the cohesion stresses in approaching the end of the contact zone, we represent the distribution of the cohesion forces as

$$\sigma_{\text{coh}}(y) = \sigma_{\text{coh}} \sin\left(\frac{\pi}{2}\,\frac{d-y}{d}\right) \quad \text{at} \quad y \leqslant d, \qquad \sigma_{\text{coh}}(y) = 0 \quad \text{at} \quad d < y < l_0.$$

We note that, the end regions of the boundary of the fictitious ellipse and the center of the contact zone are acted upon by the greatest possible cohesion stresses $\sigma_{\text{coh}} = \sigma_{\text{coh}}(0)$.

Solving the problem with the framework of the modified elastoplastic analog of the Griffith problem for an ellipse, we determine the normal displacements and stresses at an arbitrary point of the plate with the fictitious ellipse, after which we construct the sufficient strength criterion. The sufficient strength criterion is a system of two equations, one of which is a condition which eliminates partial overlapping of the upper and lower surfaces of the contact zone, and the other is a deformation criterion of critical opening of the ellipse. For a certain set of initial parameters, we find a combination of the critical external compressive load $\sigma_\infty^*/\sigma_{\text{coh}}$ and the critical lengths of the prefracture zone $\Delta_1^* = \Delta^*/l_0$ and the contact zone $d_1^* = d^*/l_0$ that satisfies the sufficient strength criterion.

2. COMPRESSION OF THE PLATE WITH THE ELLIPTIC HOLE

2.1. Stresses and Normal Displacements at an Arbitrary Point of the Plate with the Elliptic Hole. According to [4], the stresses and displacements at an arbitrary point of a plate with an elliptic hole are expressed in terms of the first and second derivatives of the Airy stress function. The Airy stress function is defined as $F = F^* + F^{**} + F^{***}$. The stress function F^* corresponds to the stress distribution in the case where the contour of the ellipse is free of stresses [4]. The stress function F^{**} describes the stress distribution for which the end regions of the contour of the fictitious ellipse are loaded by compressive stresses $-\sigma_{\text{coh}}$. The stress function F^{***} describes the stress distribution for which the central regions of the contour of the fictitious ellipse corresponding to the contact zone are loaded by the compressive stresses $-\sigma_{\text{coh}}(y)$. Inclusion of the additional stress function F^{***} in the Airy stress function provides the occurrence of a contact zone under compression. The additional stresses decrease rapidly with distance from the elliptic hole. The boundary conditions for the stress functions F^*, F^{**}, and F^{***} are derived from the conditions on the contour of the ellipse expressed in terms of stresses [3, 4].

Thus, the problem breaks up into three problems, by solving which it is possible to find the stress functions F^*, F^{**}, and F^{***} and, hence, the stresses and displacements at an arbitrary point of the plate with the deformed contour of the ellipse.

We introduce the elliptic coordinates [3–5]

$$x_1 = \frac{x}{l_0} = \frac{c_f}{l_0} \sinh u \cos v, \qquad y_1 = \frac{y}{l_0} = \frac{c_f}{l_0} \cosh u \sin v, \tag{1}$$

where $2c_f$ is the interfocal distance of the fictitious ellipse such that

$$\frac{c_f^2}{l_0^2} = (1+\Delta_1)^2\Big(1 - \frac{\rho_f}{l_0}\,\frac{1}{1+\Delta_1}\Big),$$

$\rho_f/l_0 = \rho_r/l_0(1+\Delta_1)$, $\Delta_1 = \Delta/l_0$, and ρ_f and ρ_r are the curvature radii at the tops of the fictitious and real ellipses, respectively. Setting c_f/l_0 in the formula defining $\Delta_1 = 0$, we obtain a formula for the interfocal distance c_r/l_0 of the real ellipse.

In view of (1), we have

$$\Big(\frac{x_1}{c_f \sinh\,(u/l_0)}\Big)^2 + \Big(\frac{y_1}{c_f \cosh\,(u/l_0)}\Big)^2 = 1.$$

Hence, the lines $u = u_0$ are ellipses.

According to [3], the stresses and normal displacements at an arbitrary point of the plate with the elliptic hole are defined by the formulas

$$\begin{aligned}
\sigma_{x_1}(x_1,y_1) &= B_1\Big(\frac{\partial B_1}{\partial u}\,\frac{\partial F}{\partial u} + B_1\,\frac{\partial^2 F}{\partial u^2} + \frac{\partial A_1}{\partial u}\,\frac{\partial F}{\partial v} + A_1\,\frac{\partial^2 F}{\partial u\,\partial v}\Big)\\
&\quad + A_1\Big(\frac{\partial B_1}{\partial v}\,\frac{\partial F}{\partial u} + B_1\,\frac{\partial^2 F}{\partial u\,\partial v} + \frac{\partial A_1}{\partial v}\,\frac{\partial F}{\partial v} + A_1\,\frac{\partial^2 F}{\partial v^2}\Big),\\
\sigma_{y_1}(x_1,y_1) &= A_1\Big(\frac{\partial A_1}{\partial u}\,\frac{\partial F}{\partial u} + A_1\,\frac{\partial^2 F}{\partial u^2} - \frac{\partial B_1}{\partial u}\,\frac{\partial F}{\partial v} - B_1\,\frac{\partial^2 F}{\partial u\,\partial v}\Big)\\
&\quad - B_1\Big(\frac{\partial A_1}{\partial v}\,\frac{\partial F}{\partial u} + A_1\,\frac{\partial^2 F}{\partial u\,\partial v} - \frac{\partial B_1}{\partial v}\,\frac{\partial F}{\partial v} - B_1\,\frac{\partial^2 F}{\partial v^2}\Big);\\
\tau_{x_1y_1}(x_1,y_1) &= A_1\Big(\frac{\partial B_1}{\partial u}\,\frac{\partial F}{\partial u} + B_1\,\frac{\partial^2 F}{\partial u^2} + \frac{\partial A_1}{\partial u}\,\frac{\partial F}{\partial v} + A_1\,\frac{\partial^2 F}{\partial u\,\partial v}\Big)\\
&\quad - B_1\Big(\frac{\partial B_1}{\partial v}\,\frac{\partial F}{\partial u} + B_1\,\frac{\partial^2 F}{\partial u\,\partial v} + \frac{\partial A_1}{\partial v}\,\frac{\partial F}{\partial v} + A_1\,\frac{\partial^2 F}{\partial v^2}\Big);
\end{aligned} \tag{2}$$

$$\frac{\xi(x_1,y_1)}{l_0} = \frac{\sigma_{\rm y}}{E}\,(1+\nu)\Big(-\frac{A_1}{\sigma_{\rm y}}\,\frac{\partial F}{\partial u} + \frac{B_1}{\sigma_{\rm y}}\,\frac{\partial F}{\partial v} - \frac{4}{4-\alpha}\,\frac{\Phi_1}{\sigma_{\rm y}}\Big). \tag{3}$$

Here

$$A_1 = \frac{1}{h^2}\,\frac{c_f}{l_0}\,\cosh u \cos v, \qquad B_1 = \frac{1}{h^2}\,\frac{c_f}{l_0}\,\sinh u \sin v,$$

$$\frac{\partial A_1}{\partial u} = \frac{\cos v \sinh u}{h_1^2 c_f/l_0}\Big(1 - \frac{2\cosh^2 u}{h_1^2}\Big), \qquad \frac{\partial A_1}{\partial v} = -\frac{\sin v \cosh u}{h_1^2 c_f/l_0}\Big(1 - \frac{2\cos^2 v}{h_1^2}\Big),$$

$$\frac{\partial B_1}{\partial u} = \frac{\sin v \cosh u}{h_1^2 c_f/l_0}\Big(1 - \frac{2\sinh^2 u}{h_1^2}\Big), \qquad \frac{\partial B_1}{\partial v} = \frac{\cos v \sinh u}{h_1^2 c_f/l_0}\Big(1 + \frac{2\sin^2 v}{h_1^2}\Big),$$

$\alpha = 2(1-\nu)$, ν is Poisson 's ratio, E is Young's elastic modulus, $h^2 = (c_f^2/l_0^2)(\sinh^2 u + \cos^2 v)$, and $h_1^2 = \sinh^2 u + \cos^2 v$ are the distortion coefficients.

We write the Airy stress function as

$$F = F^* + F^{**} + F^{***} = \Phi_0 + x_1\Phi_1,$$

where $\Phi_0 = \Phi_0(x_1,y_1) = \Phi_0^* + \Phi_0^{**} + \Phi_0^{***}$; $\Phi_1 = \Phi_1(x_1,y_1) = \Phi_1^* + \Phi_1^{**} + \Phi_1^{***}$ ($\Delta\Phi_0 = 0$, $\Delta\Phi_1 = 0$), $F^* = \Phi_0^* + x_1\Phi_1^*$, $F^{**} = \Phi_0^{**} + x_1\Phi_1^{**}$, and $F^{***} = \Phi_0^{***} + x_1\Phi_1^{***}$.

The stress function F^* is determined with accuracy up to the sign in a paper [5], which gives a solution of the problem of tension of a plate with a hole in the form of an ellipse. Because, the present work deals with compression rather than tension, the stress function F^* has the opposite sign to that in [5]. Then, for compression conditions, the formulas for the stress function F^*, its partial derivatives, and the function Φ_1^* become

$$F^* = -\frac{\sigma_\infty}{8}\,\frac{c_f^2}{l_0^2}\Big(1 + \cosh\,2u + 2A^*u + 2C^*\,\mathrm{e}^{-u}\sinh\,u + (-\cosh\,2u - 1 + 2B^*\,\mathrm{e}^{-2u} + 2C^*\,\mathrm{e}^{-u}\sinh\,u)\cos 2v\Big),$$

$$\frac{\partial F^*}{\partial u} = -\frac{\sigma_\infty}{4}\,\frac{c_f^2}{l_0^2}\Big(\sinh\,2u + A^* + C^*\,\mathrm{e}^{-2u} + (-\sinh\,2u - 2B^*\,\mathrm{e}^{-2u} + C^*\,\mathrm{e}^{-2u})\cos 2v\Big),$$

$$\frac{\partial F^*}{\partial v} = \frac{\sigma_\infty}{4}\,\frac{c_f^2}{l_0^2}\Big(-\cosh\,2u - 1 + 2B^*\,\mathrm{e}^{-2u} + 2C^*\,\mathrm{e}^{-u}\sinh\,u\Big)\sin 2v,$$

$$\frac{\partial^2 F^*}{\partial u\,\partial v} = \frac{\sigma_\infty^{(1)}}{2}\,\frac{c_f^2}{l_0^2}\Big(-\sinh\,2u - 2B^*\,\mathrm{e}^{-2u} + C^*\,\mathrm{e}^{-2u}\Big)\sin 2v, \tag{4}$$

$$\frac{\partial^2 F^*}{\partial u^2} = -\frac{\sigma_\infty^{(1)}}{2}\,\frac{c_f^2}{l_0^2}\Big(\cosh\,2u - C^*\,\mathrm{e}^{-2u} + (-\cosh\,2u + 2B^*\,\mathrm{e}^{-2u} - C^*\,\mathrm{e}^{-2u})\cos 2v\Big),$$

$$\frac{\partial^2 F^*}{\partial v^2} = -\frac{\sigma_\infty^{(1)}}{2}\,\frac{c_f^2}{l_0^2}\Big(-\cosh\,2u - 1 + 2B^*\,\mathrm{e}^{-2u} + 2C^*\,\mathrm{e}^{-u}\sinh\,u\Big)\cos 2v,$$

$$\Phi_1^* = -\frac{\sigma_\infty}{2}\,\frac{c_f}{l_0}\Big(\sinh\,u + C^*\,\mathrm{e}^{-u}\Big)\cos v,$$

where A^*, B^*, and C^* are some constants. The addition of the terms containing these constants to the right sides of formulas (4) provides a rapid reduction in the stresses with distance from the hole. It should be noted that, despite the change in the sign of the stress function F^* from [5], the formulas defining the constants A^*, B^*, and C^* are the same for the cases of compression and tension:

$$A^* = -1 - \cosh\,2u_0, \qquad B^* = \frac{1}{2}\,\mathrm{e}^{2u_0} + \frac{3}{4} - \frac{1}{4}\,\mathrm{e}^{4u_0}, \qquad C^* = 1 + \mathrm{e}^{2u_0},$$

because the stress function F^* for the cases of tension and compression satisfies the boundary conditions

$$\frac{\partial F^*}{\partial u} = 0, \qquad \frac{\partial F^*}{\partial v} = 0.$$

Let us determine the stress function F^{**}, its partial derivatives, and the function Φ_1^{**} under compression conditions. For this, in formulas (4), we replace the superscript "$*$" by the superscript "$**$" and the external compressive stress $-\sigma_\infty$ by the cohesion stress $-\sigma_{\rm coh}$. If, in formulas (4), the superscript "$*$" is replaced with the superscript "$**$" and the external compressive stress $-\sigma_\infty$ is replaced with the distribution $-\sigma_{\rm coh}(y)$, it is possible to determine the stress function F^{***}, its partial derivatives, and the function Φ_1^{***} under compression conditions.

The boundary conditions for the stress function F^{**} are determined from the equilibrium condition for the forces in the x_1 and y_1 directions

$$X^{**} = -\Big(\frac{\partial F^{**}}{\partial y_1}\Big)_{v=v_2} + \Big(\frac{\partial F^{**}}{\partial y_1}\Big)_{v=v_1}, \qquad Y^{**} = -\Big(\frac{\partial F^{**}}{\partial x_1}\Big)_{v=v_2} + \Big(\frac{\partial F^{**}}{\partial x_1}\Big)_{v=v_1} = 0 \tag{5}$$

and the condition of zero moment of forces M^{**} about the point $x_1 = 0$, $y_1 = 0$

$$M^{**} = \Big(x_1\,\frac{\partial F^{**}}{\partial x_1} + y_1\,\frac{\partial F^{**}}{\partial y_1} - F^{**}\Big)_{v=v_2} - \Big(x_1\,\frac{\partial F^{**}}{\partial x_1} + y_1\,\frac{\partial F^{**}}{\partial y_1} - F^{**}\Big)_{v=v_1} = 0. \tag{6}$$

Here the partial derivatives $\partial F^{**}/\partial x_1$ and $\partial F^{**}/\partial y_1$ are determined from the formulas

$$\frac{\partial F^{**}}{\partial x_1} = \frac{c_f/l_0}{h^2}\Big(\cosh\,u_0\cos v\,\frac{\partial F^{**}}{\partial u} - \sinh\,u_0\sin v\,\frac{\partial F^{**}}{\partial v}\Big),$$

$$\frac{\partial F^{**}}{\partial y_1} = \frac{c_f/l_0}{h^2}\Big(\sinh\,u_0\sin v\,\frac{\partial F^{**}}{\partial u} - \cosh\,u_0\cos v\,\frac{\partial F^{**}}{\partial v}\Big).$$

In equalities (5), the quantities X^{**} and $Y^{**}=0$ are projections of the forces caused by the compressive cohesion stresses $\sigma_{\rm coh}$ acting on the end of the contour of the fictitious ellipse for $u=u_0$ and $v_1\leqslant v\leqslant v_2$ $(1\leqslant y_1\leqslant 1+\Delta_1)$. The quantity X^{**} is determined from the formula

$$X^{**}=-\sigma_{\rm coh}\int\limits_{v_1}^{v_2} d\,\frac{\tilde l}{l_0}=-\frac{c_f}{l_0}\,\sigma_{\rm coh}\int\limits_{v_1}^{v_2}\sqrt{\sinh^2 u_0+\cos^2 v}\,dv, \tag{7}$$

where $\sin v_1=1/[(c_f/l_0)\cosh u_0]=1/(1+\Delta_1)$, $\sin v_2=(1+\Delta_1)/[(c_f/l_0)\cosh u_0]=1$, $\cos v_1<0$, $\tilde l/l_0$ is the dimensionless length of the contour $u=u_0$ at $v_1\leqslant v\leqslant v_2$. Since, in the vicinity of the top of the fictitious ellipse for small values of Δ_1, the value of $\cos v$ is close to zero, formula (7) becomes

$$X^{**}\approx-(c_f/l_0)\sigma_{\rm coh}\sinh u_0\,(\pi/2-v_1).$$

It is necessary to note that all terms on both sides of equality (5) contain the multiplier $-(c_f/l_0)\sigma_{\rm coh}$. To simplify the calculations, we divide $\partial F^{**}/\partial x_1$, $\partial F^{**}/\partial y_1$, and X^{**} by this quantity. Then, we shall consider the quantities

$$\frac{\partial\,(F^{**})'}{\partial x_1}=-\frac{l_0\sigma_{\rm coh}}{c_f}\,\frac{\partial F^{**}}{\partial x_1},\qquad \frac{\partial\,(F^{**})'}{\partial y_1}=-\frac{l_0\sigma_{\rm coh}}{c_f}\,\frac{\partial F^{**}}{\partial y_1},$$

$$(X^{**})'=-X^{**}\,\frac{l_0\sigma_{\rm coh}}{c_f}\approx\sinh u_0\,(\pi/2-v_1).$$

Formula (6) can be simplified if we take into account that, for $Y^{**}=0$, we have $(\partial(F^{**})'/\partial x_1)_{v=v_2}=(\partial(F^{**})'/\partial x_1)_{v=v_1}$. Then, the condition of zero moment of the forces M^{**} about the point $x_1=0$, $y_1=0$ is written as

$$\Big(x_1\Big|_{v=v_2}-x_1\Big|_{v=v_1}\Big)\,\frac{\partial\,(F^{**})'}{\partial x_1}\Big|_{v=v_2}+\Big(y_1\Big|_{v=v_2}-y_1\Big|_{v=v_1}\Big)\,\frac{\partial\,(F^{**})'}{\partial y_1}\Big|_{v=v_2}$$
$$-\,y_1\Big|_{v=v_1}(X^{**})'+(F^{**})'\Big|_{v=v_1}-(F^{**})'\Big|_{v=v_2}=0. \tag{8}$$

In view of boundary conditions (5) and (6), for the stress function F^{**}, we obtain the system of equations

$$\begin{aligned} A_{01}A^{**}+B_{01}B^{**}+C_{01}C^{**}+D_{01}&=(X^{**})',\\ A_{02}A^{**}+B_{02}B^{**}+C_{02}C^{**}+D_{02}&=0,\\ A_{03}A^{**}+B_{03}B^{**}+C_{03}C^{**}+D_{03}&=0,\end{aligned} \tag{9}$$

where

$$A_{01}=\frac{\sinh u_0\sin v_1}{4(\sinh^2 u_0+\cos^2 v_1)}-\frac{1}{4\sinh u_0},$$

$$B_{01}=-\frac{{\rm e}^{-2u_0}}{2}\Big(\frac{\sinh u_0\sin v_1\cos 2v_1+\cosh u_0\cos v_1\sin 2v_1}{\sinh^2 u_0+\cos^2 v_1}+\frac{1}{\sinh u_0}\Big),$$

$$C_{01}=\frac{\cos^2 v_1\sin v_1\,{\rm e}^{-u_0}\sinh u_0}{2(\sinh^2 u_0+\cos^2 v_1)}\,({\rm e}^{-u_0}-2\cosh u_0),$$

$$D_{01}=\frac{2\sinh u_0\sin^3 v_1\sinh 2u_0+\cosh u_0\cos v_1\sin 2v_1(\cosh 2u_0+1)}{4(\sinh^2 u_0+\cos^2 v_1)}-\cosh u_0,$$

$$A_{02}=\cosh u_0\cos v_1,\qquad B_{02}=2\,{\rm e}^{-2u_0}(\sinh u_0\sin v_1\sin 2v_1-\cosh u_0\cos v_1\cos 2v_1),$$

$$C_{02}=2\,{\rm e}^{-u_0}\cos v_1\,(2\sinh^2 u_0\sin^2 v_1+\cosh u_0\cos^2 v_1\,{\rm e}^{-u_0}),\qquad D_{02}=0,$$

$$A_{03}=\coth u_0\,(1-\sin v_1)/4,$$

$$B_{03}={\rm e}^{-2u_0}(2\coth u_0-\coth u_0\sin v_1+\cos 2v_1+1)/4,\qquad C_{03}={\rm e}^{-u_0}\sinh u_0\cos^2 v_1/2,$$

$$D_{03}=\cosh^2 u_0-\cosh u_0\sin v_1\,((X^{**})'+\cosh u_0)+(1+\cosh 2u_0)\sin^2 v_1/4-(\cosh 2u_0+1)/4.$$

The first two equations of system (9) are defined by the equilibrium conditions for the forces in the x_1 and y_1 directions (5), and the last equation is defined by the condition of zero moment of the forces about the point $x_1 = 0$, $y_1 = 0$ (8).

Solving system (9), we find the constants A^{**}, B^{**}, and C^{**}:

$$A^{**} = \frac{(X^{**})'}{A_{01}} - \frac{B^{**}B_{01}}{A_{01}} - \frac{C^{**}C_{01}}{A_{01}} - \frac{D_{01}}{A_{01}},$$

$$B^{**} = \frac{A_{02}D_{01} - A_{01}D_{02} - A_{02}(X^{**})' - C^{**}(C_{02}A_{01} - A_{02}C_{01})}{A_{01}B_{02} - A_{02}B_{01}},$$

$$C^{**} = [(A_{03}D_{01} - A_{03}(X^{**})' - A_{01}D_{03})(A_{01}B_{02} - A_{02}B_{01})$$

$$- (A_{02}D_{01} - A_{02}(X^{**})' - A_{01}D_{02})(A_{01}B_{03} - A_{03}B_{01})]$$

$$\Big/[(A_{01}C_{03} - A_{03}C_{01})(A_{01}B_{02} - A_{02}B_{01}) - (A_{01}C_{02} - A_{02}C_{01})(A_{01}B_{03} - A_{03}B_{01})].$$

Having determined the constants A^{**}, B^{**}, and C^{**}, we can calculate the stress function F^{**}.

The boundary conditions for the stress function F^{***} determined from the equilibrium condition for the forces in the x_1 and y_1 directions are written as

$$X^{***} = \int\limits_{v=v_3}^{v=v_4} dX^{***} = -\int\limits_{v=v_3}^{v=v_4} d\Big(\frac{\partial F^{***}}{\partial y_1}\Big) = -\Big(\frac{\partial F^{***}}{\partial y_1}\Big)_{v=v_4} + \Big(\frac{\partial F^{***}}{\partial y_1}\Big)_{v=v_3},$$

$$\Big(\frac{\partial F^{***}}{\partial x_1}\Big)_{v=v_4} = \Big(\frac{\partial F^{***}}{\partial x_1}\Big)_{v=v_3}. \qquad (10)$$

Here X^{***} is a projection of the force due to the presence of the compressive stresses $-\sigma_{\rm coh}(y)$ acting on the central region of the contour of the fictitious ellipse for $u = u_0$ and $v_3 \leqslant v \leqslant v_4$ ($0 \leqslant y_1 \leqslant d_1$, $d_1 = d/l_0$), where

$$\sin v_3 = 0, \quad \sin v_4 = \frac{d_1}{(c_f/l_0)\cosh u_0} = \frac{d_1}{1+\Delta_1}, \quad \cos v_3 = -1, \quad \cos v_4 < 0.$$

It should be noted that the right and left sides of expressions (10) contain the distribution $-\sigma_{\rm coh}(y)$. We divide $\partial F^{***}/\partial x_1$, $\partial F^{***}/\partial y_1$, and X^{***} by the quantity $-(c_f/l_0)\sigma_{\rm coh}(y)$. In the following, we consider the quantities

$$\frac{\partial\,(F^{***})'}{\partial x_1} = -\frac{l_0\sigma_{\rm coh}(y)}{c_f}\,\frac{\partial F^{***}}{\partial x_1}, \qquad \frac{\partial (F^{***})'}{\partial y_1} = -\frac{l_0\sigma_{\rm coh}(y)}{c_f}\,\frac{\partial F^{***}}{\partial y_1},$$

$$(X^{***})' = -X^{***}\,\frac{l_0\sigma_{\rm coh}(y)}{c_f}. \qquad (11)$$

The quantity $(X^{***})'$ is defined by the formula

$$(X^{***})' = \int\limits_{v_3}^{v_4} \sqrt{\sinh^2 u_0 + \cos^2 v}\; dv = \cosh\, u_0 \int\limits_{v_3}^{v_4} \sqrt{1 - k^2\sin^2 v}\; dv = \cosh\, u_0 E(v_4, k),$$

where

$$E(v,k) = \frac{2v}{\pi}\,E + \sin v\cos v\,\Big[\frac{k^2}{4} + \frac{k^4}{8}\Big(\frac{1}{4}\,\sin^2 v + \frac{3}{8}\Big) + \frac{3k^6}{48}\Big(\frac{1}{6}\,\sin^4 v + \frac{5}{24}\,\sin^2 v + \frac{15}{48}\Big) + \ldots\Big],$$

$$E = \frac{\pi}{2}\Big(1 - \frac{k^2}{4} - \frac{3k^4}{64} - \ldots\Big)$$

is an elliptic integral of the second kind and the full elliptic integral, respectively; $k = (\cosh\, u_0)^{-1}$.

In view of (10) and (11), the condition of zero moment of the forces M^{***} about the point $x_1 = 0$, $y_1 = 0$ is written as

$$\Big(x_1\Big|_{v=v_4} - x_1\Big|_{v=v_3}\Big) \frac{\partial\,(F^{***})'}{\partial x_1}\Big|_{v=v_3} + \Big(y_1\Big|_{v=v_4} - y_1\Big|_{v=v_3}\Big) \frac{\partial\,(F^{***})'}{\partial y_1}\Big|_{v=v_3}$$

$$- y_1\Big|_{v=v_4}(X^{***})' + (F^{***})'\Big|_{v=v_3} - (F^{***})'\Big|_{v=v_4} = 0. \qquad (12)$$

After appropriate calculations, the boundary conditions for the stress function F^{***} are written as the system of three equations

$$\begin{aligned} A_{11}A^{***} + B_{11}B^{***} + C_{11}C^{***} + D_{11} &= (X^{***})', \\ A_{22}A^{***} + B_{22}B^{***} + C_{22}C^{***} + D_{22} &= 0, \\ A_{33}A^{***} + B_{33}B^{***} + C_{33}C^{***} + D_{33} &= 0, \end{aligned} \qquad (13)$$

where

$$A_{11} = -\frac{\sinh\, u_0 \sin v_4}{4(\sinh^2 u_0 + \cos^2 v_4)}, \qquad B_{11} = \frac{\mathrm{e}^{-2u_0}}{2}\Big(\frac{\sinh\, u_0 \sin v_4 \cos 2v_4 + \cosh\, u_0 \cos v_4 \sin 2v_4}{\sinh^2 u_0 + \cos^2 v_4}\Big),$$

$$C_{11} = -\frac{\cos^2 v_4 \sin v_4 \sinh\, u_0}{2(\sinh^2 u_0 + \cos^2 v_4)}, \qquad D_{11} = -\frac{\sinh\, u_0 \sinh\, 2u_0 \sin^3 v_4 + \cosh^3 u_0 \cos v_4 \sin 2v_4}{2(\sinh^2 u_0 + \cos^2 v_4)},$$

$$A_{22} = \frac{\cosh\, u_0 \cos v_4}{4(\sinh^2 u_0 + \cos^2 v_4)} + \frac{\cosh\, u_0}{4(\sinh^2 u_0 + 1)},$$

$$B_{22} = \frac{\mathrm{e}^{-2u_0}}{2}\Big(\frac{\sinh\, u_0 \sin v_4 \sin 2v_4 - \cosh\, u_0 \cos v_4 \cos 2v_4}{\sinh^2 u_0 + \cos^2 v_4} - \frac{\cosh\, u_0}{\sinh^2 u_0 + 1}\Big),$$

$$C_{22} = \frac{\mathrm{e}^{-u_0} \cos v_4}{2}\Big(\frac{2 \sinh^2 u_0 \sin^2 v_4 + \cosh\, u_0 \cos^2 v_4\, \mathrm{e}^{-u_0}}{\sinh^2 u_0 + \cos^2 v_4} + \frac{\cosh\, u_0\, \mathrm{e}^{-u_0}}{\sinh^2 u_0 + 1}\Big), \qquad D_{22} = 0,$$

$$A_{33} = -\frac{\sinh\, 2u_0}{8(\sinh^2 u_0 + 1)}(\cos v_4 + 1), \qquad B_{33} = \mathrm{e}^{-2u_0}\Big(\frac{\sinh\, 2u_0(\cos v_4 + 1)}{4(\sinh^2 u_0 + 1)} + \frac{\sin^2 v_4}{2}\Big),$$

$$C_{33} = \frac{\mathrm{e}^{-u_0} \sinh\, u_0 \sin^2 v_4}{2} - \frac{\sinh\, 2u_0(\cos v_4 + 1)\, \mathrm{e}^{-2u_0}}{4(\sinh^2 u_0 + 1)},$$

$$D_{33} = -\cosh\, u_0 \sin v_4 (X^{***})' - (1 + \cosh\, 2u_0)\,\frac{\sin^2 v_4}{4}.$$

The first two equations of system (13) are defined by the equilibrium conditions for the forces in the x_1 and y_1 directions (10), and the last equation is defined by the condition of zero moment of forces about the point $x_1 = 0$, $y_1 = 0$ (12).

Solving system (13), we find the constants A^{***}, B^{***}, and C^{***}:

$$A^{***} = \frac{(X^{***})'}{A_{11}} - \frac{B^{***}B_{11}}{A_{11}} - \frac{C^{***}C_{11}}{A_{11}} - \frac{D_{11}}{A_{11}},$$

$$B^{***} = \frac{A_{22}D_{11} - A_{11}D_{22} - A_{22}(X^{***})' - C^{***}(C_{22}A_{11} - A_{22}C_{11})}{A_{11}B_{22} - A_{22}B_{11}},$$

$$C^{***} = [(A_{33}D_{11} - A_{33}(X^{***})' - A_{11}D_{33})(A_{11}B_{22} - A_{22}B_{11}) - (A_{22}D_{11} - A_{22}(X^{***})' - A_{11}D_{22})(A_{11}B_{33} - A_{33}B_{11})]$$

$$\Big/ [(A_{11}C_{33} - A_{33}C_{11})(A_{11}B_{22} - A_{22}B_{11}) - (A_{11}C_{22} - A_{22}C_{11})(A_{11}B_{33} - A_{33}B_{11})].$$

After the constants A^{***}, B^{***}, and C^{***} are determined, it is possible to calculate the stress function F^{***}.

All components of the Airy function $F = F^{*} + F^{**} + F^{***}$ are found. The stresses and normal displacement at an arbitrary point of the plate with the deformed contour of the fictitious ellipse are calculated by formulas (2) and (3).

We next construct a sufficient strength criterion and calculate the critical fracture parameters.

2.2. Critical Fracture Parameters of the Plate with the Deformed Ellipse Contour under Compression. As noted above, the atoms of the central pair are the first to come in contact; hence, with a further increase in the contact area, they will be subjected to the greatest external compressive load. Therefore, it is necessary to choose a compressive load at which the displacement $\xi(x_0, 0)$ of the point with the coordinates $u = u_0$ and $v = -\pi$ does not exceed $x_0 = (c_f/l_0)\sinh u_0 = \sqrt{(1+\Delta_1)^2 - (c_f/l_0)^2}$, which will make it possible to eliminate partial overlapping of the surfaces of the contact zone. The condition for the optimum load that rules out partial overlapping of the surfaces of the contact zone is written as

$$\xi(x_0, 0) = -x_0. \tag{14}$$

However, condition (14) does not guarantee that the plate with the deformed contour of the elliptic hole is in equilibrium. Therefore, we consider the deformation criterion of critical opening of the ellipse [5]

$$2\xi(x_{00}, l_0) = h_{\rm t}, \tag{15}$$

where $2\xi(x_{00}, l_0)$ is the opening of the fictitious ellipse at the top of the real ellipse, $x_{00}/l_0 = \sqrt{(1-(1+\Delta_1)^{-2})((1+\Delta_1)^2 - (c_f/l_0)^2)}$, $h_{\rm t}$ is the critical opening of the real ellipse at the top. The quantity $h_{\rm t}$ is expressed in terms of the transverse size of the plastic zone at the top of the real ellipse and the relative elongation of the plastic material:

$$h_{\rm y} = 2h(\varepsilon_{\rm y} - \varepsilon_0).$$

Here $\varepsilon_0 = \sigma_{\rm y}/E$ is the ultimate elongation of the elastic material; $\varepsilon_{\rm y}$ is the ultimate elongation of the plastic material. We recall that the stresses $\sigma_{\rm y}$ are understood as the material cohesion forces $\sigma_{\rm coh}$.

Thus, we have constructed a sufficient strength criterion representing a system of two equations, one of which is a condition which eliminates partial overlapping of the upper and lower surfaces of the contact zone [see formula (14)], and the other is a deformation criterion of critical opening of the ellipse [see formula (15)]. Criterion (14), (15) is a deformation-strength criterion.

The displacements $\xi(x_0, 0)$ in formula (14) are defined by formula (3). Performing appropriate rearrangements of equality (14), we obtain the ultimate load

$$p = \frac{\sigma_{\rm y}}{\sigma_\infty}(\Delta_1, d_1, \varepsilon_0, \rho_r/l_0) = \frac{N_1}{N_2}, \tag{16}$$

at which the central region of the contour of the elliptic hole is deflected but partial overlapping of the surfaces of the contact zone is absent. In formula (16),

$$N_1 = \frac{\cosh u_0}{4(\sinh^2 u_0 + 1)}\Big(A^{*} + 2C^{*}\,{\rm e}^{-2u_0} - 2B^{*}\,{\rm e}^{-2u_0}\Big) + \frac{2}{4-\alpha}\Big(\sinh u_0 + C^{*}\,{\rm e}^{-u_0}\Big),$$

$$N_2 = -\frac{x_0(l_0/c_f)}{\varepsilon_0(1+\nu)} + \frac{\cosh u_0}{4(\sinh^2 u_0 + 1)}\Big(-A^{**} - A^{***} + 2\,{\rm e}^{-2u_0}(-C^{**} - C^{***} + B^{***} + B^{**})\Big)$$

$$+ \frac{2}{4-\alpha}\,(-C^{**} - C^{***})\,{\rm e}^{-u_0}.$$

The opening of the fictitious ellipse at the top of the real ellipse $2\xi(x_{00}, l_0)$ in the deformation criterion (15) is defined by formula (3). The transverse size of the plastic zone at the top of the real ellipse h in criterion (15) can be estimated using the Mises plasticity criterion. In the case of a plane stress state, the Mises plasticity criterion has the form

$$3\Big(\frac{\sigma_{x_1} - \sigma_{y_1}}{2}\Big)^2 + 3\tau_{x_1y_1}^2 + \Big(\frac{\sigma_{x_1} + \sigma_{y_1}}{2}\Big)^2 = \sigma_{\rm y}^2. \tag{17}$$

TABLE 1

Critical Fracture Parameters

Material	ε_y	ε_0	ρ_r/l_0	p^*	Δ_1^*	d_1^*	h
Annealed copper having nanostructure	0.45220	0.00220	0.001	10.538	0.001	0.998	0.006
			0.005	4.318	0.031	0.957	0.038
Carbon steel (0.1% C)	0.33124	0.00095	0.001	4.383	0.015	0.993	0.037
			0.005	2.894	0.115	0.987	0.094
Carbon steel (0.3% C)	0.24176	0.00157	0.001	3.834	0.006	0.995	0.050
			0.005	2.691	0.084	0.966	0.112

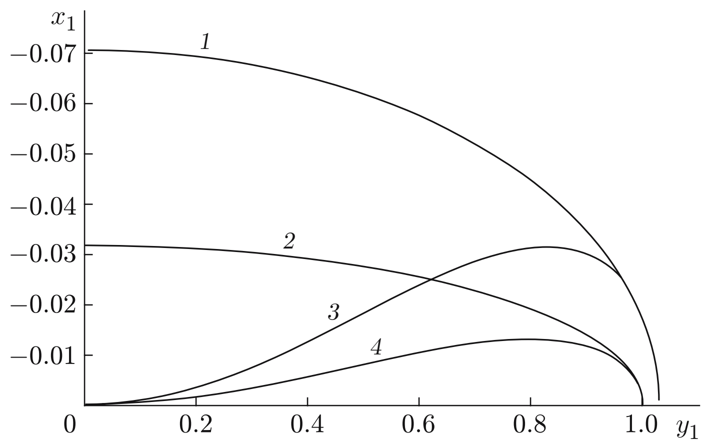

Fig. 3. Initial shapes (1 and 2) and final shapes (3 and 4) of the boundary of the elliptic hole during compression of plates of annealed copper having nanostructure: 1) $\rho_r/l_0 = 0.005$; 2) $\rho_r/l_0 = 0.001$.

Substituting formulas (2) of the stress at an arbitrary point of the plate with a real elliptic hole into (17) (ρ_r/l_0 is the curvature radius at the top of the real ellipse; $c_r/l_0 = \sqrt{1-\rho_r/l_0}$) for $y_1 = 1$ ($v = v_1$), we have the equation

$$g(p, \rho_r/l_0, h) = 0, \tag{18}$$

from which we can estimate the transverse size of the plastic zone $2h$ for the ultimate load p calculated by (16) and the specified value of ρ_r/l_0. A detailed description of the calculation of the transverse size of the plastic region formed during tension of a plate with an elliptic hole is given in [3].

Let us calculate the critical fracture parameters: the critical value of the external compressive load p^* and the critical lengths of the prefracture zone Δ_1^* and the contact zone d_1^*. The critical fracture parameters can be obtained from the sufficient strength criterion (14), (15). We first determine the safe load p. The parameters ε_0, ε_y, and ρ_r/l_0 will be considered specified. Searching through the pairs $(\Delta_1^{(i)}, d_1^{(j)})$ ($\Delta_1^{(i+1)} = \Delta_1^{(i)} + t$ and $d_1^{(j+1)} = d_1^{(j)} + t$, where $i = 0, \ldots, n$ and $j = 0, \ldots, m$) for $t = 0.0005$, $\Delta_1^{(0)} = 0.0001$, and $d_1^{(0)} = 0.0001$, we calculate p by formula (16). We fix only such combinations of parameters $(\Delta_1^{(i)}, d_1^{(j)})$ for which $p > 1$. It is necessary to determine whether any of these combinations satisfies the deformation criterion of critical opening of the ellipse (16). For $y_1 = 1$ ($v = v_1$), we calculate the opening $2\xi(x_{00}, l_0)/l_0$ of the fictitious ellipse at the top of the real ellipse by formula (3). Using formulas (2) and (18), we estimate the transverse size $2h$ of the plastic zone. The deformation criterion of critical opening of the ellipse (16) is written as

$$2\xi(x_{00}, l_0) - h_y = f(\Delta_1^{(i)}, d_1^{(j)}, p, h, \varepsilon_0, \varepsilon_y, \rho_r/l_0) \approx 0. \tag{19}$$

According to (19), we choose only such a combination of the parameters $\Delta_1^{(i)}$, $d_1^{(j)}$, p, h, ε_0, ε_y, ρ_r/l_0 for which the quantity $f(\Delta_1^{(i)}, d_1^{(j)}, p, h, \varepsilon_0, \varepsilon_y, \rho_r/l_0)$ takes a minimum value close to zero. The parameters $p^* = p$, $\Delta_1^* = \Delta_1^{(i)}$, and $d_1^* = d_1^{(j)}$ are the critical fracture parameters for which there is no partial overlapping of the upper and lower surfaces of the contact zone and the plate with the deformed contour of the ellipse is in the ultimate equilibrium state.

Table 1 gives the critical fracture parameters p^*, Δ_1^*, and d_1^* for some plastic materials and numerical estimates of the transverse size of the plastic zone h. The values of the parameters ε_0 and $\varepsilon_{\rm y}$ are taken from [6]. According to the data given in the table, the more plastic the material, the smaller remote load is required to maintain the equilibrium configuration. Plates with holes having a greater curvature radius at the top should be compressed by greater remote loads to maintain the equilibrium configuration. The action of such loads leads to the occurrence of more extensive plastic zones, which, in turn, influences the length of the contact zone. The size of the plastic zone characterizes the transverse size of the contact zone: the larger the size of the plastic zone, the smaller the length of the contact zone.

Figure 3 shows the deformation of the boundaries of ellipses with various curvature radii for annealed copper having nanostructure. The boundaries of ellipses with curvature radii $\rho_r/l_0 = 0.005$ (curve 1) and 0.001 (curve 2) become curves 3 and 4, respectively.

CONCLUSIONS

The substantially nonlinear problem was solved by reducing the elastoplastic problem defined for a modified Leonov–Panasyuk–Dugdale model to an elastoplastic analog of the Griffith problem by using the Goodier and Kanninen model. This considerably simplified the problem, making to possible to qualitatively and quantitatively describe the compression of a plate with an elliptic hole with the formation of a contact zone.

This work was supported by the Russian Foundation for Basic Research (Grant No. 07-01-00163) and the program of the Presidium of the Russian Academy of Sciences (Grant No. 11.16).

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